UDC 004.052.32

SYNTHESIS OF OPTIMAL ONE-STEP T-DIAGNOSABLE GRAPHS FOR A DETERMINISTIC ASYMMETRICAL SYSTEM-LEVEL DIAGNOSIS MODEL

M.B. KRAMARENKO
ELEKS, Ltd.

Abstract *- Deterministic asymmetrical system-level diagnosis model is considered assuming connected graphs, permanent faults and complete t-diagnosis. Diagnosis model properties are investigated. It is shown that any one-step t-diagnosable graph always consists of at least 2·(n - 1) edges, where 1 < t ≤ n - 1. The notion of optimal one-step t-diagnosable graphs is defined for 1 < t ≤ n - 1. It is shown that optimal one-step t-diagnosable graphs with 2·(n - 1) edges can be synthetized based on undirected spanning trees by replacement of each unoriented edge with two oppositely directed edges. It is proven that synthetized optimal graphs are always one-step t-diagnosable.*

The practical value of the considered deterministic asymmetrical system-level diagnosis model is the possibility to describe timer-based tests assuming that a faulty timer always masks the faults of tested units but never distorts the state of fault-free tested units.

Анотація *– Для детермінованої асиметричної моделі діагностування на системному рівні (system-level diagnosis) розв'язано задачу синтезу оптимальних графів, що паралельно t- діагностуються за умови стійких відмов. Практична користь проаналізованої моделі полягає в її здатності описувати таймерні тести за умови, що контролюючий таймер при власній відмові завжди маскує відмову модуля, що ним перевіряється, але в жодному випадку не спотворює технічний стан працездатного модуля, що ним перевіряється.*

Аннотация *– Для детерминированной ассиметричной модели диагностирования на системном уровне (system-level diagnosis) решена задача синтеза оптимальных параллельно t-диагностируемых графов для случая устойчивых отказов. Практическая ценность проанализированной модели состоит в ее способности описывать таймерные тесты, предполагая, что контролирующий таймер при собственном отказе всегда маскирует отказ проверяемого им модуля, но никогда не искажает техническое состояние работоспособного проверяемого им модуля.*

Introduction

A diagnosable system S consists of a set of units $U=\{u_0,u_1,\dots,u_n\},|U|=n$, where each unit is capable to test at least one other unit but can't test itself [1-5]. The tests assignment is represented by a weighted directed graph $G(U,E)$ without self-loops, where the set of vertices represents the set of units belonging to the diagnosable system S. Each edge $e_{ij}\in E$ exists if and only if both vertices u_i,u_j participate in the same test. The edge $e_{ij}\in E$ direction means that the vertex u_i is a testing vertex and u_j is a tested vertex. The weight (or label) a_{ij} of each edge $e_{ij}\in E$ is a Boolean variable which is equal to 0 if and only if the testing vertex u_i decided that the tested vertex u_j is fault-free and equal to 1 otherwise. A matrix containing all test outcomes is called a syndrome $\delta=\{a_{ij}:(i,j)\in E\}$. The syndrome where all test outcomes are "0" is denoted by $\sum_0$. The syndrome is decoded by a dedicated fault-free unit (global arbiter) which is responsible for system diagnosis based on test outcomes only. The set $F \subseteq U, 0 \le |F| \le t, t < n$ is a set of faulty vertices.

The test outcome depends on actual testing and tested vertex states. Thus each diagnosis model can be defined by a tuple $< a_{gg}\ a_{gb}\ a_{bg}\ a_{bb} >$ [4]:

a_{gg} – fault-free vertex decision about the state of a fault-free vertex;

a_{gb} – fault-free vertex decision about the state of a faulty vertex;

a_{bg} – faulty vertex decision about the state of a fault-free vertex;

a_{bb} – faulty vertex decision about the state of a faulty vertex.

It's clear that $a_{gg} = 0; a_{gb} = 1$ and $a_{bg}, a_{bb} \in \{0,1.-\}$, where the symbol "-" means an arbitrary (0 or 1) test result.

For example, the well-known PMC-model [2] is defined by the tuple **0 1 - -**, and BGM-model [3] – by the tuple **0 1 - 1**.

The practical value of **0 1 0 0** diagnosis model is the possibility to describe timer-based tests assuming that a faulty timer always masks the faults of tested units but never distorts the state of fault-free tested units [4].

The **0 1 0 0** diagnosis model was studied in [4, 5]. The necessary conditions of t-diagnosis were found in [4]. Characterization problem for one-step t-diagnosis was studied in [5]. The synthesis of optimal t-diagnosable graphs has not been considered yet.

The goal of this paper is synthesis of the optimal one-step t-diagnosable graphs assuming connected graphs, permanent faults and complete t-diagnosis, where $t < n, n = |U|$.

I. Analysis

The direct consequence of the **0 1 0 0** diagnosis model definition is that $a_{bg}, a_{bb} \notin \{-\}, a_{bg} \ne a_{gb}$, thus this model is a deterministic asymmetrical system-level diagnosis model. For given vertex u, the following vertex sets can be defined:

– $\Gamma^{+1}(u)$ as a vertex set $u, v \in U$ where $(u,v) \in E$;

– $\Gamma^{-1}(u)$ as a vertex set $u, v \in U$ where $(v,u) \in E$.

Similarly for some vertex set $U_1, U_1 \subseteq U$, the following vertex sets can be defined [5]:

– $B(U_1)$ as a vertex set $\bigcup \Gamma^{-1}(u_i)$ for each $u_i \in U_1$;

– $D(U_1)$ as a vertex set $\bigcup \Gamma^{+1}(u_i)$ for each $u_i \in U_1$;

– $\Gamma^{-1}(U_1)$ as a vertex set $B(U_1) - U_1$;

– $\Gamma^{+1}(U_1)$ as a vertex set $D(U_1) - U_1$;

– $C(U_1)$ as a vertex set $u_i \in U_1$, where $(u_j, u_i) \in E, u_j \in U - U_1$.

Test validity for **0 1 0 0** diagnosis model is shown in fig. 1.

From the definition of **0 1 0 0** diagnosis model follows:

Lemma 1. Any "one-labeled" edge exists if and only if it is incident from a fault-free vertex and incident to a faulty vertex.

From the definition of **0 1 0 0** diagnosis model, the following is proved by induction:

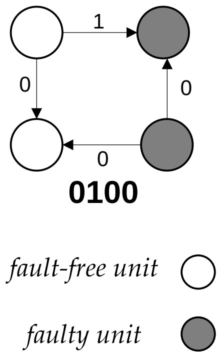

Fig. 1. Possible test results for **0 1 0 0** diagnosis model.

Lemma 2. Any outgoing path with "zero-labeled" edges originating from a fault-free vertex consists of fault-free vertices only.

Lemma 3. Any path with "zero-labeled" edges incoming to a faulty unit consists of faulty vertices only.

From the definition of **0 1 0 0** diagnosis model and Lemma 1 follow:

Lemma 4. If any vertex $u \in U$ has a "one-labeled' incident from (or outgoing) edge then all "zero-labeled" edges incident from this vertex are incident to fault-free vertices only.

Lemma 5. If any vertex $u \in U$ has a "one-labeled" incoming edge then all zero-labeled edges incoming to this vertex outgo from faulty vertices only.

From Lemmas 2- 5 follow:

Lemma 6. Any "zero-labeled" edge path which outgoes from a vertex with a "one-labeled" outgoing edge consists of fault-free vertices only.

Lemma 7. Any "zero-labeled" edge path which incomes to a vertex with at least one "one-labeled" incoming edge consists of faulty vertices only.

Lemma 8. No vertex can have "one-labeled" incoming and outgoing edges simultaneously.

From the definition of **0 1 0 0** diagnosis model and Lemma 1 follows:

Lemma 9. The syndrome $\sum_0$ exists if and only if there are no edges incident from fault-free vertices and incident to faulty vertices.

Graph $G(U,E)$ is said to be *one-step t-diagnosable* if one application of the set of tests is sufficient to identify all faulty vertices $F \subseteq U$ provided that $|F| \le t$ [1-5].

The necessary conditions of one-step t-diagnosis are determined by the following theorem.

Theorem 1[4]. The necessary condition of one-step diagnosis is $1 \le t \le n-1$. If $1 \le t \le n-1$ then there exists a one-step t-diagnosable graph $G(U,E)$.

The idea behind the proof directly follows from considering the conditions of existence of at least one "one-labeled" edge in the complete graph $G(U,E)$. It should be noted that the conditions of $t=0$ and $t=n$ in any graph $G(U,E)$ due to Lemma 9 necessarily form syndrome $\sum_0$, which means the inability to distinguish between two different technical states of the diagnosable system S without additional assumptions.

Corollary 1. Graph $G(U,E)$ is one-step t-diagnosable if at least one vertex is faulty.

Theorem 2[5]. Graph $G(U,E)$ is one-step t-diagnosable if and only if for any vertex subset $U_1 \subseteq U, |U| \le t$, two conditions are satisfied:

1. $\Gamma^{-1}(U_1) \neq \varnothing$.
2. For any vertex $u_i \in U_1 - C(U_1)$, there exists a path incoming to at least one vertex belonging to $C(U_1)$, which comes across vertices from set U_1 only.

Corollary 2. For any vertex $u \in U$ of one-step t-diagnosable graph $G(U,E)$, $\Gamma^{-1}(u) \neq \varnothing$. Proof directly follows from the first condition of Theorem 2 considering $U_1 \subset U, |U_1| = 1$. In such case any subset U_1 shrinks to some vertex $u \in U$ and thus $\Gamma^{-1}(u) \neq \varnothing$.

Corollary 3. Any one-step t-diagnosable graph $G(U,E)$ contains a cycle.

Corollary 4. For any vertex $u \in U$ of one-step t-diagnosable graph $G(U,E)$, $\Gamma^{+1}(u) \neq \varnothing$, where $1 < t \le n-1$.

Proof directly follows from the fact that if $\Gamma^{+1}(u) \neq \varnothing$, it is impossible to create any path incoming to at least one vertex from $C(U_1)$ which comes across vertices belonging to vertex set U_1 only, $|U_1| > 1$.

Corollary 5. The necessary condition for adding any new vertex u_i to one-step t-diagnosable graph $G(U,E)$, where $1 < t \le n-1$, is existence of at least two oppositely directed edges incident to u_i.

Let's consider some existing one-step t-diagnosable graph $G(U,E)$ and try to add to it some new vertex u_i. From Corollary 2 follows that $\Gamma^{-1}(u) \neq \varnothing$. Let's assume that there exists some edge incident from some other vertex u_j and incident to the vertex u_i (see fig. 2). From Corollary 4 follows that the vertex u_i must be incident from at least one edge. Such edge must be incident to the vertex u_j because otherwise for given subset $U_1 = \{u_i, u_j\}, |U_1| = 2$, it is impossible to construct an incoming path to $C(U_1) = \{u_j\}$, which comes across vertices belonging to the set U_1 only (vertex u_i in this case).

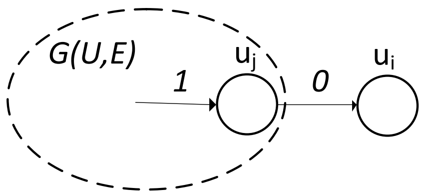

Fig. 2. Violation of the Second Condition for Theorem 2

From Corollary 5 directly follows:

Corollary 6. Any one-step t-diagnosable graph $G(U,E)$ consists of at least $2 \cdot (n-1)$ edges, where $1 < t \le n-1$.

The following definition of optimality is a direct consequence of Corollary 6:

Definition 1. A one-step t-diagnosable graph $G(U,E)$ is optimal for $1 < t \le n-1$ if $|E| = 2 \cdot (n-1)$.

II. Synthesis

Consider a spanning tree of undirected connected graph. The set of edges of such tree has minimal cardinality. Let's use this property for optimal one-step t-diagnosable graphs synthesis.

Let's define $K_{p,q}$ as an undirected bipartite graph with two parts consisting of p and q vertices respectively. Thereafter a directed graph generated form $K_{1,n-1}$ by replacing each unoriented edge with two oppositely directed edges is denoted by $Z_{1,n-1}$-graph and named as a "star-graph" here.

Theorem 3. A $Z_{1,n-1}$-graph is always one-step t-diagnosable.

Indeed, consider the case with fault-free central vertex u of the $Z_{1,n-1}$-graph. In such case all "one-labeled" edges incident from the vertex u will be incident to fault vertices and all "zero-labeled" edges will be incident to fault-free vertices.

Respectively, consider the case with fault of the central vertex u. In such case the $Z_{1,n-1}$-graph will contain at least one "one-labeled" edge incident to the central vertex u and incident from the fault-free peripheral vertex. Thus the central vertex u will be diagnosed as faulty and all faulty peripheral vertices will be one-step diagnosed using Lemma 5. Moreover, each "one-labeled" edge will be incident from a fault-free peripheral vertex.

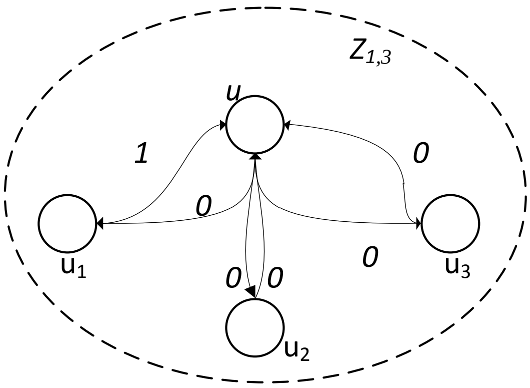

Fig. 3. Graph $Z_{1,3}$ with fault-free peripheral vertex u_1 and faulty vertices u, u_2, u_3

Similarly, consider an $L(U)$-graph generated from a given undirected elementary path by replacing each unoriented edge with a pair of oppositely directed edges. For such graphs from Lemmas 1-6, the following theorem is derived:

Theorem 4. An $L(U)$-graph is always one-step t-diagnosable.

The direct consequence of Theorem 1 is existence of at least one "one-labeled" edge within such graph. Thus an $L(U)$-graph is always one-step t-diagnosable using Lemmas 1-6.

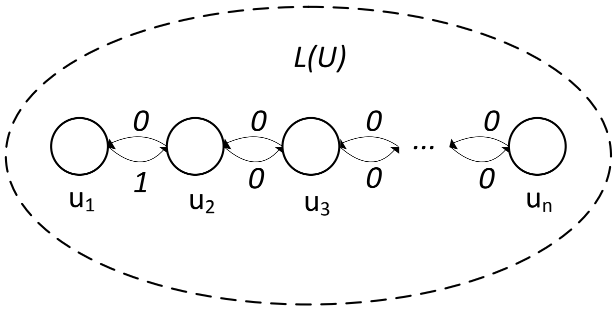

Fig. 4. $L(U)$-graph with fault-free vertex u_1 and faulty vertices $u_2, u_3, \ldots u_n$

Suggested approach is applicable to generation of a one-step t-diagnosable graph from any undirected spanning tree. Let's denote such type of graphs by *TK(U)*. For such graphs from Lemmas 1-6, similarly to Theorems 3, 4 the following theorem is derived:

Theorem 5. A *TK(U)*-graph is always one-step t-diagnosable.

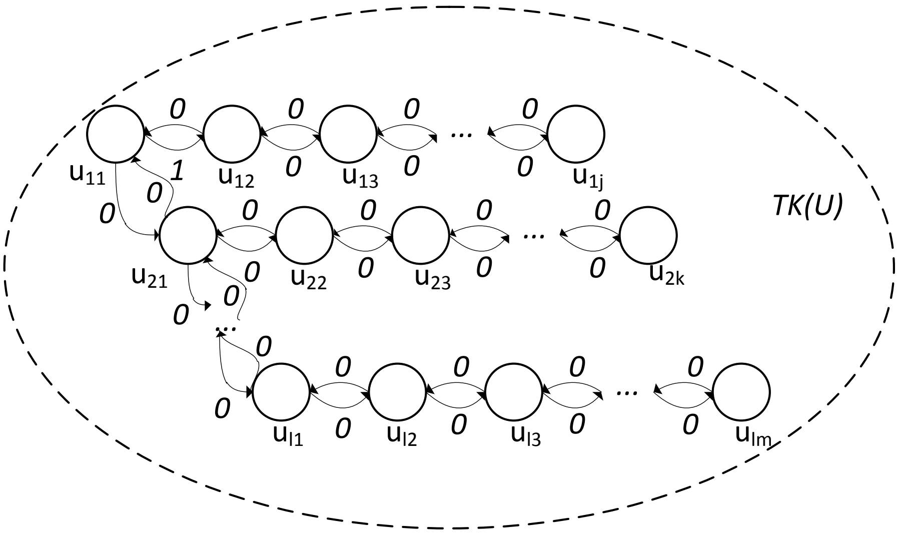

Fig. 5. *TK(U)*-graph with faulty vertices $u_{12}, u_{13}, \ldots, u_{1j}$

Let's show by induction that $Z_{1,n-1}$ -, *L(U)*- and *TK(U)*-graphs satisfy Theorem 2. Firstly, consider a minimal *L(U)*-graph containing two vertices. It's obvious that condition 1 of Theorem 2 is satisfied for $t = 1$. Condition 2 of Theorem 2 is also satisfied because any set $U_1 - C(U_1), |U_1| = 1$, shrinks to the empty set.

Secondly, consider an *L(U)*-graph containing three vertices (see fig. 6).

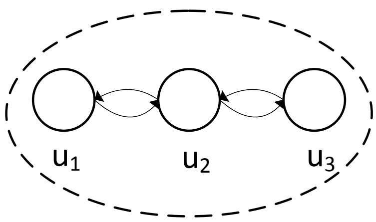

Fig. 6. $L(U)$-graph which coincides with graph $Z_{1,2}$

Conditions 1 and 2 of Theorem 2 are satisfied for $t=1$. For $t=2$, consider the vertex set $U_1 \subset U, |U_1|=2$, which consists of vertices u_1 and u_2. In such case vertex u_3 comprises the set $\Gamma^{-1}(U_1) \neq \varnothing$, and there exists a path from vertex u_1 to vertex u_2 which comprises the set $C(U_1)$. Similarly, satisfaction of Conditions 1 and 2 of Theorem 2 can be proven for the vertex set $U_1 \subset U, |U_1|=2$, which consists of vertices u_2 and u_3. For set $U_1 \subset U, |U_1|=2$, which consists of vertices u_1 and u_3, Conditions 1 and 2 of Theorem 2 are satisfied because set $U_1 - C(U_1)$ is empty.

Finally, consider an $L(U)$-graph where $U=\{u_0, u_1, \ldots, u_{n-1}\}$ and assume that it satisfies Conditions 1 and 2 of Theorem 2. Let's add to it some vertex u_n, connect this vertex to vertex u_{n-1} with a pair of oppositely directed edges and assume that the resulting graph does not satisfy Conditions 1 and 2 of Theorem 2. Then the original graph does not satisfy Conditions 1 and 2 of Theorem 2. Since a contradiction arises, this completes the proof that the $L(U)$-graph satisfies Conditions 1 and 2 of Theorem 2. The same approach can be used for $Z_{1,n-1}$ - and $TK(U)$-graphs.

The direct consequence of the proposed synthesis algorithm of one-step t-diagnosable optimal graphs is:

Corollary 7. $Z_{1,n-1}$-, $L(U)$- and $TK(U)$-graphs always contain $2 \cdot (n-1)$ edges.

From Corollary 7 directly follows:

Theorem 6. $Z_{1,n-1}$ -, $L(U)$- and $TK(U)$-graphs are one-step t-diagnosable optimal graphs for $1 < t \leq n-1$.

Moreover, it is possible to show the impossibility of deleting any edge from $Z_{1,n-1}$ -, $L(U)$- and $TK(U)$-graphs. Indeed, consider an $L(U)$-graph and remove from it any edge, e.g. the edge e_{ij} incident from the vertex u_i and incident to the vertex u_j. Consider the case when edge e_{ij} is "zero-labeled". As a result, the same syndrome will be generated for two different states of vertices: $u_i, u_j \in F$, or $u_i, u_j \in U-F$. Example of such a case is shown in fig. 7.

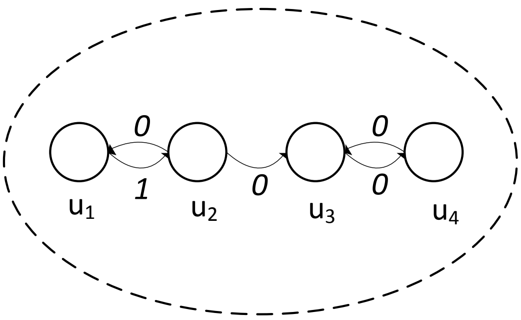

Fig. 7. Non-diagnosable L(U)-graph with removed edge

It's clear that the vertex u_1 is fault-free and the vertex u_2 is faulty, but vertices u_3 and u_4 could be faulty as well as fault-free.

Thus, it's impossible to remove any edge form this *L*(*U*)-graph, so it is a one-step *t*-diagnosable optimal graph for $1 < t \le n-1$. Optimality of $Z_{1,n-1}$ - and *TK*(*U*)- graphs can be shown in a similar manner.

Conclusion

Deterministic asymmetrical system-level **0 1 0 0** diagnosis model is considered for connected graphs assuming permanent faults and complete *t*-diagnosis.

Optimal one-step t-diagnosable graphs with 2·(*n* - 1) edges are synthetized for 1<t≤n-1. It is shown that the synthetized graphs are always one-step t-diagnosable.

The paper outcomes could be used for fault-tolerant computer systems design.

References

1. *Mostafa Abd-El-Barr* Design and analysis of reliable and fault-tolerant computer systems. – Imperial College Press, 2007. – 440 p.
2. *Preparata F.P., Metze G., Chien R.T.* On the Connection Assignment Problem of Diagnosable Systems // IEEE Trans, on Electronic Computers. – 1967. – Vol. EC-16, No. 6.– P. 848-854.
3. *Barsi F., Grandoni F., Maestrini P.* A theory of diagnosability of digital systems // IEEE Trans Computers. – 1976. – Vol. C-25. –P. 585-593.
4. *Крамаренко М.Б.* Модели диагностирования отказов параллельной вычислительной системы. // Электронное моделирование. – 1989. – № 3. – С 60-65.
5. *Радойчевски В.Ц., Шалаев А.Я.* Параллельная диагностируемость модульных систем при централизованной дешифрации синдрома // Электронное моделирование. – 1992. – № 1. – С. 57—63.